Production Heterogeneity in Collective Labor Supply Models with Children

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Abstract

Children's welfare is at the center of many welfare reforms. A key goal for policy-makers is to evaluate the costs and benefits of such reforms. The main challenge lies in that the outcome of interest, children's welfare, is unobservable. To address this issue, I consider a collective labor supply model with children where adult members have preferences over their own leisure, expenditures, and children's welfare. I show that the model nonparametrically partially identifies the impacts of parental inputs on children's welfare in panel data. I then propose a novel estimation strategy that accommodates measurement error and can be used to efficiently construct valid confidence sets. Using Dutch data on couples with children, I investigate

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the structure of the expected production technology and how it varies with household characteristics. I find that the production of children's welfare is characterized by decreasing returns to scale and large heterogeneity across household types.

JEL Classification: D11, D12, D13, C51, C63.

1 Introduction

Parents have a significant impact on children's welfare through a variety of individual and collective investments such as time spent with children and expenditure on children. These decisions are influenced by parents' preferences for leisure, consumption, and children's welfare, as well as by their time and budget constraints. This paper provides a framework to analyze the complex decision-making process that underlies parental investments in children's welfare and underscores the importance of addressing disparities in parenting skills across different demographics to mitigate achievement gaps among children.

It is widely recognized that the unitary model, which assumes that household members preferences can be represented by a single household utility function, is inappropriate for analyzing household data (see e.g., Fortin and Lacroix (1997) and Browning and Chiappori (1998)). In an effort to provide a proper foundation to analyze household behavior, Chiappori (1988, 1992) suggested a collective model in which household members have distinct preferences and whose allocations are the result of a Pareto efficient bargaining process. This framework has proven to be empirically successful at rationalizing household decision-making (e.g., Cherchye and Vermeulen, 2008) and understanding power dynamics within the household (e.g., Cherchye, De Rock and Vermeulen, 2011).

The collective model was later extended to household production by Apps and Rees (1997) and Chiappori (1997).[1] Both papers show that the distribution of

[1]In a similar model to the one of Chiappori (1988), Apps and Rees (1988) already include home production to analyze the effects of taxation on welfare.

resources within the household, known as the sharing rule, cannot be identified when the output from production is unobservable such as with children's welfare. However, they show that the sharing rule can be identified up to a function of wages when the production technology exhibits constant returns to scale (CRS). Perhaps because of the serious identification problem that arises with home production, the literature has since maintained the constant returns to scale assumption (Blundell, Chiappori and Meghir, 2005; Chiappori and Ekeland, 2009; Cherchye, De Rock and Vermeulen, 2012; Hubner, 2023).

Once the household is recognized as a collection of individuals, it becomes possible to understand the impacts of a policy targeted at a specific individual. For example, the distribution of resources within the household has been shown to have significant impacts on children's welfare as early as Thomas (1990). Motivated by this "targeted" view, Blundell, Chiappori and Meghir (2005) extended the collective framework to caring parents where children's welfare is treated as a public good in parents preferences and produced via time investment and expenditure on children. Their ideas were then brought to the data by Cherchye, De Rock and Vermeulen (2012) using a unique panel data set containing information on time use and expenditures.

As Apps and Rees (1997) acknowledged, however, there is a need to verify the empirical validity of the assumptions required for identification. Indeed, it is now well-recognized that children's welfare is crucial in weighing the costs and benefits of social programs.[2] For example, policy-makers may be interested in determining optimal cash assistance for single mothers such as to increase children's skills development (Mullins, 2022), cash transfers to poor families such as to improve children's life outcomes (Aizer et al., 2016), and parenting programs for disadvantaged households such as to narrow children's achievement gaps (Waldfogel and Washbrook, 2011).[3] Clearly, returns on investment in children play a key role in

[2]See Aizer, Hoynes and Lleras-Muney (2022) for a review of social programs in the United States and the importance of considering children's welfare.

[3]See Kalil (2014) for a review of the literature on the importance of parenting on child development and Shah and Gennetian (2024) for a recent survey of the literature on the impacts of cash transfers on families with children.

determining optimal cash transfers, and parenting programs are mostly useful if households differ in their effectiveness at developing children's skills.

The empirical literature has largely investigated these questions by using measurable quantities such as children's cognitive and noncognitive skills.[4] Although this approach has been fruitful, it requires rich data sets whose measures of skills are often noisy (Cunha and Heckman, 2008; Cunha, Heckman and Schennach, 2010). In contrast, the collective approach proposed by Blundell, Chiappori and Meghir (2005) treats children's welfare as an unobservable, thus facilitating policy evaluations when data are limited. I believe the collective model is a natural framework that can complement costs and benefits analyses of programs directed at families, but I am also sympathetic to Apps and Rees (1997)'s reservations regarding conditions for identification.

My first contribution is to assess the plausibility of CRS within the collective model with children. Consistent with previous work, I show that the production function is nonparameterically identified under CRS in cross-sectional data, and nonidentified when CRS is relaxed. Nevertheless, I show that the production function can be nonparameterically partially identified in panel data when it exhibits decreasing returns to scale, even with unrestricted heterogeneity. Indeed, the panel structure of the data gives additional implications to shape constraints on the production technology and preferences. Since household members have preferences over children's welfare, these constraints generate restrictions on the production function.

My partial identification argument uses nonparametric revealed preference conditions implied by the collective labor supply model with children.[5] The revealed preference analysis of the collective model was developed in the original work of Cherchye, De Rock and Vermeulen (2007, 2011). I extend their former characterization by further incorporating household production.[6] Interestingly, I show

[4]See Del Boca, Flinn and Wiswall (2014) for early work on the estimation of the production function for child quality in the unitary model.

[5]See also Dunbar, Lewbel and Pendakur (2013) for a collective model with children that does not require the share of resources allocated to children to be known.

[6]See Varian (1984) for early work on a revealed preference analysis of production.

that the collective model implies profit maximizing behavior in the production of children's welfare, thus giving rise to a two-step argument. In a first step, I show that restrictions on the distribution of returns to scale eliminate production functions from the set of profit-maximizing production functions. In a second step, I show that revealed preference conditions can restrict the distribution of returns to scale.

My second contribution is to propose a novel estimation strategy to analyze the collective model with unrestricted preference and production heterogeneity. To this end, I use the framework developed by Aguiar and Kashaev (2021) which provides a tractable approach to make statistical testing and inference in partially identified models defined by shape constraints.[7] The main challenge faced in applying their framework is that collective models tend to be highly nonlinear. This poses a nontrivial computational problem as existing implementations only work well for models defined by linear constraints.[8] The reason is that the method involves integrating out the set of solutions of the model, but nonlinearities make it hard to sample from the feasible space.

I solve this practical limitation by proposing a blocked Gibbs sampler that allows direct sampling from the feasible space even when the latter does not define a polytope.[9] Intuitively, the idea is to break down the feasible space into conditional convex polytopes for which closed-form bounds on the support of the latent variables can be obtained. I observe that my methodology may prove useful for other collective models (Cherchye, Demuynck and De Rock, 2011, d'Aspremont and Dos Santos Ferreira, 2019, Cherchye et al., 2020) and, more generally, other models that contain nonlinear constraints.

[7] Their framework builds on the Entropic Latent Variable Integration via Simulation (ELVIS) methodology developed by Schennach (2014). Intuitively, ELVIS can be viewed as a generalization of the method of simulated moments (McFadden, 1989; Pakes and Pollard, 1989).

[8] For example, Aguiar and Kashaev (2021) consider the collective exponential discounting model of Adams et al. (2014) but only test necessary conditions to simplify the implementation. Gauthier (2025) considers a model of price search, but assumes a quasilinear specification in the application that alleviates the computational burden.

[9] Direct sampling generally uses a Hit-and-Run algorithm that requires the feasible space to define a (convex) polytope. See Aguiar and Kashaev (2021) for an application to models defined by shape constraints and Demuynck (2021) for an application to models defined by the Generalized Axiom of Revealed Preference (GARP) as introduced by Varian (1982).

My third contribution is to empirically investigate the impact of time inputs on children's welfare in households with children. To do so, I use the Longitudinal Internet Studies for the Social sciences (LISS) panel data from Cherchye, De Rock and Vermeulen (2012). I first investigate the average impacts of parental inputs on children's welfare. I find that a doubling of all inputs increases children's welfare by about 35% on average, hence providing empirical evidence against the CRS assumption. Examining each input separately, I find that a doubling in time spent on childcare would increase children's welfare by about 10% for fathers and 14% for mothers, while a doubling in expenditure on children would increase children's welfare by about 9%.

Second, I examine how the average production technology varies with household characteristics. I find that higher education levels increase the impacts of parental time inputs on children's welfare by about 6% for mothers and about 2% for fathers. Further, I find that the education level of the mother has a positive spillover effect of about 3% on the impacts of time inputs by fathers. Interestingly, for households that are not homeowners, the impacts of parental time inputs is about 7% and 11% lower for mothers and fathers, respectively. Taken together, my results show that children from disadvantaged households, whose parents often have lower education levels and are not homeowners, are significantly worse off.

The approach taken in this paper relates to the research program initiated by Blundell, Browning and Crawford (2003, 2007, 2008). In this series of papers, the authors show how to combine revealed preference restrictions with additional information (e.g., expansion paths) to improve bounds on cost-of-living indices and demand responses. I also exploit revealed preference restrictions, but use them to learn about the production of children's welfare within a collective model. I take advantage of recent developments in the partial identification literature to impose these restrictions in a cohesive statistical framework. Moreover, my method allows for unrestricted individual heterogeneity, an important feature for the application.

The econometric framework taken in this paper views individuals within the household as random draws from a fixed utility distribution. Given the panel structure of the data, this amounts to a random utility model where a household preferences are drawn in the first period and kept constant. Hence, the approach relates to the burgeoning nonparametric random utility literature started by Kitamura and Stoye (2018).[10] Contrary to my method, their approach considers a unitary model and only requires data on cross-sectional distributions of choices. Subsequent work include Hubner (2020), Deb et al. (2023), Kashaev et al. (2023), Lazzati, Quah and Shirai (2023), and Tebaldi, Torgovitsky and Yang (2023).

The paper is organized as follows. Section 2 describes the collective model and characterizes its implications. Section 3 studies identification in the model. Section 4 presents the empirical specification. Section 5 presents the estimation strategy. Section 6 presents the data set used in the application. Section 7 presents the empirical results. Section 8 concludes. The Appendix contains proofs that are not in the main text and the Gibbs sampler.

2 Household Model

This section presents the environment considered in the paper, the collective model, and its empirical implications.

2.1 Environment

I consider households with two adults ($i = 1, 2$) and children. I assume that parents care about their children and incorporate this feature in the model by treating children's welfare as a public good. The preferences of each adult household member over leisure, expenditures and children's welfare are represented by a utility function U^i that is continuous, increasing, and concave.

At every observation $t \in \mathcal{T} = \{1, 2, \ldots T\}$, adult household members spend their time on leisure l_t^i, market work b_t^i, and childcare h_t^i such that the following time constraint is satisfied:

$$l_t^i + b_t^i + h_t^i = \tau,$$

[10] The theoretical ideas were put forth by McFadden and Richter (1990) and McFadden (2005).

where τ is the total amount of time available in a time period. Parents use time spent on childcare and expenditure on children (c_t) to produce children's welfare. The relationship between parents inputs and children's welfare is formalized through the production function

$$W_t \equiv F(h_t^1, h_t^2, c_t)e^{\epsilon_t},$$

where $\epsilon_t \in \mathbb{R}$ represents a productivity shock. Each household member receives a wage w_t^i per unit of market work. As such, the budget constraint is given by

$$q_t + Q_t + c_t = y_t + w_t^1 b_t^1 + w_t^2 b_t^2,$$

where $q_t \in \mathbb{R}_+$ represents expenditure on private goods, $Q_t \in \mathbb{R}_+$ represents expenditure on public goods, and $y_t > 0$ represents nonlabor income.

Since private expenditure cannot be used simultaneously by both household members, it has to be split in some way between them.

Definition 1. For every observation $t \in \mathcal{T}$, I say that $q_t^i \in \mathbb{R}_+$, $i \in \{1, 2\}$, represent personalized private expenditures of each household member if $\sum_{i=1}^{2} q_t^i = q_t$.

Household members get utility from their share of private expenditure such that their preferences depend on leisure, private expenditure, public expenditure, and children's welfare. In what follows, I assume that private expenditure of each household member is observed to match the data available in the application. However, the results can be generalized to the case where only total private expenditure is observed.

Let $\mathcal{U}$ denote the set of continuous, increasing, and concave utility functions and $\mathcal{W}$ the set of continuous, increasing, and concave in (h_t^1, h_t^2, c_t) production functions. Each household $j \in \mathcal{J}$ is characterized by a triple (U_j^1, U_j^2, W_j), drawn i.i.d. from a joint distribution on $\mathcal{U} \times \mathcal{U} \times \mathcal{W}$. Conditional on (U_j^1, U_j^2, W_j), a household data set $D_j := \{(q_{jt}^i, Q_{jt}, c_{jt}, b_{jt}^i, h_{jt}^i, w_{jt}^i)_{i=1}^2\}_{t \in \mathcal{T}}$ is a draw from some distribution, which allow for arbitrary correlation across time. For notational simplicity, I do not explicitly write the household subscript j on variables unless it is relevant. The next subsection formalizes the relationship between the data and the abstract notion of household through the lenses of the collective model.

2.2 Collective Model

I follow Chiappori (1988, 1992) and assume that household members choose an intrahousehold allocation that is Pareto efficient. This choice is motivated by the observation that Pareto efficiency is a minimal condition for optimal resource allocation (and hence, rationality) in a group setting. Hence, for every observation $t \in \mathcal{T}$, the household picks an intrahousehold allocation that solves

$$\max_{(l^1,l^2,h^1,h^2,q^1,q^2,Q,c)\in\mathbb{R}^2_+\times\mathbb{R}^2_{++}\times\mathbb{R}^2_+\times\mathbb{R}_+\times\mathbb{R}_{++}} \mu^1_t U^1(l^1, q^1, Q, W) + \mu^2_t U^2(l^2, q^2, Q, W), \quad (1)$$

subject to satisfying the constraints

$$\begin{aligned}(q^1 + q^2) + Q + c &= y_t + w^1_t b^1 + w^2_t b^2 \\ l^i + b^i + h^i &= \tau \ (i = 1, 2) \\ W &= F(h^1, h^2, c)e^{\epsilon_t}.\end{aligned}$$

where $\mu^i_t > 0$ denote the bargaining power of household member i. Note that the model makes no assumption on the underlying process by which the Pareto efficient allocation is achieved. That is, the weights μ^i_t result from some black box bargaining process that takes place within the household.[11]

I propose a natural notion of collective rationalizability based on the household maximization problem.

Definition 2. Let D be a data set. The model (1) rationalizes the data if there exist concave utility functions U^i, a concave production function F, and productivity shocks ϵ_t such that the first-order conditions of the model are satisfied.

[11]Although Nash equilibria are not always Pareto efficient, the black box bargaining process could be a (Pareto efficient) Nash equilibrium. Indeed, since married couples effectively play a repeated game, an appeal to folk theorems provide some intuitive motivation for the idea that the Pareto efficient allocation is a (cooperative) Nash equilibrium.

This definition states that the model rationalizes the data if there are latent model parameters that satisfy the first-order conditions.[12] Since household members utility functions are concave and the budget set is linear, the first-order conditions exhaust the empirical content of the model.

2.3 Characterization

This section derives restrictions on the data implied by the model. First, I define a few notions that will be useful for the characterization of the model.

Definition 3. Let D be a data set. For every observation $t \in \mathcal{T}$, I say that $\mathcal{P}_t^i \in \mathbb{R}_{++}$, $i \in \{1, 2\}$, represent personalized (or Lindahl) prices for public expenditure of each household member if $\sum_{i=1}^{2} \mathcal{P}_t^i = 1$.

Definition 4. Let D be a data set. For every observation $t \in \mathcal{T}$, I say that $P_t^i \in \mathbb{R}_{++}$, $i \in \{1, 2\}$, represent personalized (or Lindahl) prices for children's welfare of each household member if $\sum_{i=1}^{2} P_t^i = P_t$.

It is worth noting that the personalized prices ($\mathcal{P}_t^i$, P_t^i) are not observed by the econometrician. Furthermore, while the price of public expenditure ($\mathcal{P}_t$) can safely be set to 1, the price of children's welfare (P_t) is unobservable as children's welfare is a nonmarket good.

I now introduce some revealed preference terminology. Let $a_{st}^i := w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t)$ and $x_t^i := (l_t^i, q_t^i, Q_t, W_t)$. I say that x_t^i is (strictly) directly revealed preferred to x_s^i if a_{st}^i $(<) \leq 0$. I say that x_t^i is revealed preferred to x_s^i if there exists a sequence $t_1, t_2, \ldots, t_m$ such that $a_{t_1 t}^i \leq 0$, $a_{t_2 t_1}^i \leq 0$, $\ldots$, $a_{t_{m-1} t_m}^i$, $a_{t_m s}^i \leq 0$. Likewise, I say that x_t^i is strictly revealed preferred to x_s^i if one of the inequalities in the sequence is strict.

Definition 5. A household member $i \in \{1, 2\}$ satisfies the Generalized Axiom of Revealed Preference (GARP) if there exist personalized prices for public expenditure $\mathcal{P}_t^i$, personalized prices for children's welfare P_t^i, and children's welfare W_t

[12] I assume that the solution is interior for simplicity of exposition, but the proofs encompass the possibility for corner solutions.

such that if x_t^i is revealed preferred to x_s^i then x_s^i is not strictly directly revealed preferred to x_t^i.

The notion of revealed preference relates the ordinal value of allocations that enter preferences of each household member to their expenditure levels. In my setup, the presence of a public good (Q) implies that the expenditure of an allocation depends on unknown personalized prices. Further, in the case of the public nonmarket good (W) neither the price or the quantity is known. Finally, it is worth observing that childcare and expenditure on children do not enter the definition of revealed preference as the preferences of a household member only depends on those through their impact on children's welfare.

Next, I introduce a profit maximization condition for the production of children's welfare.

Definition 6. A household satisfies profit maximization if there exist personalized prices for children's welfare P_t, a production function F, and productivity shocks ϵ_t such that $P_t F(h_t^1, h_t^2, c_t)e^{\epsilon_t} - w_t^1 h_t^1 - w_t^2 h_t^2 - c_t \geq P_t F(h^1, h^2, c)e^{\epsilon_t} - w_t^1 h^1 - w_t^2 h^2 - c$ for all inputs (h^1, h^2, c) and all $t \in \mathcal{T}$.

This definition states that the household allocates time and expenditure on children in a way that no alternative allocation would yield a higher profit, where profit is the household's valuation of children's welfare minus the opportunity cost of producing it. The following condition mirrors GARP but is formulated for household profit maximization.

Definition 7. A household satisfies the Generalized Axiom of Profit Maximization (GAPM) if there exist personalized prices for children's welfare $P_t > 0$, numbers $F_t > 0$, and productivity shocks ϵ_t such that $P_t F_t e^{\epsilon_t} - w_t^1 h_t^1 - w_t^2 h_t^2 - c_t \geq P_t F_s e^{\epsilon_t} - w_t^1 h_s^1 - w_t^2 h_s^2 - c_s$ for all $s, t \in \mathcal{T}$.

It is standard to show that a household satisfies GAPM if and only if it behaves as a profit-maximizer (Varian, 1984). I take this equivalence as given and do not delve into the construction of a well-behaved production function from GAPM. The following result provides equivalent characterizations of the model.

Theorem 1. *Let D be a given data set. The following conditions are equivalent:*

(i) The household model (1) *rationalizes the data.*

(ii) There exist personalized prices for public expenditure $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, personalized prices for children's welfare $P_t^i > 0$, numbers U^i, λ_t^i, W_t, $F_t > 0$ and productivity shocks ϵ_t such that for all $s, t \in \mathcal{T}$ and each adult member $i \in \{1, 2\}$

$$U_s^i - U_t^i \leq \lambda_t^i \left[w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t) \right],$$
$$F_s - F_t \leq \frac{w_t^1}{P_t e^{\epsilon_t}}(h_s^1 - h_t^1) + \frac{w_t^2}{P_t e^{\epsilon_t}}(h_s^1 - h_t^2) + \frac{1}{P_t e^{\epsilon_t}}(c_s - c_t),$$

where $W_t = F_t e^{\epsilon_t}$ for all $t \in \mathcal{T}$.

(iii) There exist personalized prices for public expenditure $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, personalized prices for children's welfare $P_t^i > 0$, numbers $F_t > 0$, and productivity shocks ϵ_t with $W_t = F_t e^{\epsilon_t}$, such that GARP holds for each adult member $i \in \{1, 2\}$ and GAPM holds.

Theorem 1 shows that the Afriat inequalities are equivalent to GARP and that those conditions must be satisfied for both household members. The latter implies that the household problem has an equivalent characterization in terms of a two-step procedure (Chiappori, 1988, 1992). That is, the solution of the household maximization problem can be viewed as the outcome of separate utility maximization problems for each adult in the household conditional on a distribution of nonlabor income.

It is interesting to note that neither the Afriat inequalities or GARP exhaust the empirical implications of the model. Indeed, the model further implies that the household satisfies GAPM, a necessary and sufficient condition for profit maximization. As such, household members increase each input in the production of children's welfare up until the point where marginal revenue equates marginal cost. Note that this profit maximizing behavior is not assumed but implied by the model.

3 Empirical Content

This section shows that the collective model informatively partially identifies the production function. Intuitively, if the production function exhibited constant returns to scale, the household would make zero profit as a firm and revenue $P_t W_t$ would equate costs $w_t^1 h_t^1 + w_t^2 h_t^2 + c_t$. In this special case, revenue would be identified and the first-order conditions would recover the log production function from its partial derivatives. I show that the household revenue from producing children's welfare is inversely proportional to its costs when the production function is homogeneous, where the factor of proportionality is given by its returns to scale. I then leverage the panel structure of the data to bound returns to scale from shape constraints on the production function and preferences.

3.1 Point Identification

In what follows, I explicitly write the household subscript $j = 1, 2, \ldots, J$ on variables and formalize the assumption that production functions are subject to Hicks-neutral productivity shocks.

Assumption 1. *The productivity shocks are Hicks-neutral such that children's welfare is given by*

$$W_{jt} = F_j(h_{jt}^1, h_{jt}^2, c_{jt}) e^{\epsilon_{jt}} \iff \log(W_{jt}) = f_j(h_{jt}^1, h_{jt}^2, c_{jt}) + \epsilon_{jt},$$

where f_j denote the natural logarithm of the production function and is assumed differentiable.

The assumption of Hicks-neutral productivity shocks is necessary to disentangle the impacts of productivity shocks and parental inputs on the production of children's welfare. The differentiability of the log production function is a technical condition that is useful for the identification argument.

Next, I suppose the production function is from the class of homogeneous production functions.

Assumption 2. *The production function is homogeneous of degree $RTS_j \in (0, 1]$.*

The homogeneity assumption allows me to relate the revenue to the cost of producing children's welfare. The following lemma formalizes this relationship.

Lemma 1. *Suppose Assumptions 1-2 hold, then $RTS_j P_{jt} W_{jt} = w^1_{jt} h^1_{jt} + w^2_{jt} h^2_{jt} + c_{jt}$ for all $t \in \mathcal{T}$.*

Lemma 1 shows that revenue is identified by cost up to the returns to scale. Intuitively, the homogeneity of the production function ensures that the total contribution of all inputs scales proportionally with output. Given that households behave as if they were profit maximizers, they equate marginal revenue to marginal cost. Multiplying these conditions by the corresponding inputs then implies that observed total cost effectively pins down revenue, up to returns to scale.

Finally, I impose a mild regularity condition that ensures sufficient variation in inputs in the cross-section.

Assumption 3. *The cross-sectional distribution of inputs $(h^1_{jt}, h^2_{jt}, c_{jt})_{t \in \mathcal{T}}$ has full support and is absolutely continuous.*

The requirement that the distribution of inputs spans its full support is necessary to identify the whole production function. For practical purposes, it is generally sufficient to identify the production function over the support of the data. In that case, the full support condition can be relaxed without any harm.

The first result shows that, if returns to scale were known, the expected log production function would be identified.

Proposition 1. *Suppose Assumptions 1-3 hold and RTS_j is known, then the expected log production function is nonparametrically identified up to scale.*

Proof. The first-order conditions with respect to inputs imply that the household equates the marginal product of factors of production to their marginal costs such that

$$\frac{\partial F_j(h^1_{jt}, h^2_{jt}, c_{jt})}{\partial h^1_{jt}} e^{\epsilon_{jt}} = \frac{w^1_{jt}}{P_{jt}}$$

$$\begin{aligned}
\frac{\partial F_j(h^1_{jt}, h^2_{jt}, c_{jt})}{\partial h^2_{jt}} e^{\epsilon_{jt}} &= \frac{w^2_{jt}}{P_{jt}} \\
\frac{\partial F_j(h^1_{jt}, h^2_{jt}, c_{jt})}{\partial c^2_{jt}} e^{\epsilon_{jt}} &= \frac{1}{P_{jt}}.
\end{aligned}$$

Divide the marginal products by W_{jt} to obtain

$$\begin{aligned}
\frac{\partial f_j(h^1_{jt}, h^2_{jt}, c_{jt})}{\partial h^1_{jt}} &= \frac{w^1_{jt}}{P_{jt}W_{jt}} \\
\frac{\partial f_j(h^1_{jt}, h^2_{jt}, c_{jt})}{\partial h^2_{jt}} &= \frac{w^2_{jt}}{P_{jt}W_{jt}} \\
\frac{\partial f_j(h^1_{jt}, h^2_{jt}, c_{jt})}{\partial c^2_{jt}} &= \frac{1}{P_{jt}W_{jt}},
\end{aligned}$$

where f_j denote the log production function. Taking the expectation gives

$$\begin{aligned}
\frac{\partial \mathbb{E}[f(h^1_t, h^2_t, c_t)]}{\partial h^1_t} &= \mathbb{E}\left[\frac{w^1_t}{P_t W_t}\right] \\
\frac{\partial \mathbb{E}[f(h^1_t, h^2_t, c_t)]}{\partial h^2_t} &= \mathbb{E}\left[\frac{w^2_t}{P_t W_t}\right] \\
\frac{\partial \mathbb{E}[f(h^1_t, h^2_t, c_t)]}{\partial c^2_t} &= \mathbb{E}\left[\frac{1}{P_t W_t}\right],
\end{aligned}$$

where I interchanged the partial derivative and integral.[13] By Lemma 1, RTS_j identifies $P_{jt}W_{jt}$ such that the expected marginal products are also identified. Next, variation in inputs in the cross-section allows us to integrate each marginal product, giving the following system of partial differential equations

$$\begin{aligned}
\int_{h^1_0}^{h^1_t} \frac{\partial \mathbb{E}[f(h^1_t, h^2_t, c_t)]}{\partial h^1_t} dh^1_t &= \mathbb{E}[f(h^1_t, h^2_t, c_t)] + C(h^2_t, c_t) \\
\int_{h^2_0}^{h^2_t} \frac{\partial \mathbb{E}[f(h^1_t, h^2_t, c_t)]}{\partial h^2_t} dh^2_t &= \mathbb{E}[f(h^1_t, h^2_t, c_t)] + C(h^1_t, c_t) \\
\int_{c_0}^{c_t} \frac{\partial \mathbb{E}[f(h^1_t, h^2_t, c_t)]}{\partial c_t} dc_t &= \mathbb{E}[f(h^1_t, h^2_t, c_t)] + C(h^1_t, h^2_t).
\end{aligned}$$

These equations can be used to recover the expected log production function up

[13] This is possible if f_j is dominated by a function whose integral is finite, a natural assumption for a production function.

to a constant:

$$\mathbb{E}[f(h_t^1, h_t^2, c_t)] = \int_{h_0^1}^{h_t^1} \frac{\partial \mathbb{E}[f(h^1, h_0^2, c_0)]}{\partial h_t^1} dh^1 + \int_{h_0^2}^{h_t^2} \frac{\partial \mathbb{E}[f(h_t^1, h^2, c_0)]}{\partial h_t^2} dh^2 + \\ + \int_{c_0}^{c_t} \frac{\partial \mathbb{E}[f(h_t^1, h_t^2, c)]}{\partial c_t} dc - C,$$

where C is a constant of integration. Hence, the expected log production function is identified over the support of the data. □

Proposition 1 states that, in principle, the model imposes enough structure to nonparameterically identify the expected log production function, provided returns to scale are known. In particular, the identification strategy could be used to nonparametrically estimate the expected log production function over the support of the data under constant returns to scale. Interestingly, note that identification does not require knowledge of children's welfare: relative changes in output with respect to changes in inputs are identified even if the absolute scale of W_{jt} is unknown. Consequently, any combination of P_{jt} and W_{jt} is observationally equivalent, as long as their product matches the total cost scaled by returns to scale.

3.2 Partial Identification

The previous section showed that the expected log production function depends both on the data and on the distribution of returns to scale. In this section, I show that the revealed preference conditions arising in panel data provide an additional source of identification absent an assumption on returns to scale.

Let $\zeta_j := \{(\mathcal{P}_{jt}^i, P_{jt}^i, W_{jt}, F_{jt}, RTS_j, \epsilon_{jt})_{i \in \{1,2\}}\}_{t \in \mathcal{T}} \in Z|\mathcal{D}$ denote the set of household-specific latent variables that enter in the definition of GAPM and GARP, where Z denote the support of the latent variables and $\mathcal{D}$ denote the support of the data. Further, let $\mathcal{P}_{Z|\mathcal{D}}$ denote the set of distributions of the latent variables conditional on the data and π_0 denote the distribution of the data. The previous discussion motivates the following definition of the identified set:

$$\Theta_0 = \big\{\mathbb{E}[f(h_t^1, h_t^2, c_t)] : \exists \mu \in \mathcal{P}_{Z|\mathcal{D}} \text{ such that } D_j \text{ is rationalized by the model for all } j \text{ and Assumptions 1-3 hold}\big\}.$$

In words, the identified set corresponds to the set of expected log production functions that arises for every distribution of the unobservables that rationalizes the data. Observe that Proposition 1 implies that each distribution of returns to scale maps to an expected log production function given the data. Hence, the identified set rules out some expected log production functions provided the model restricts the expected returns to scale to a strict subset of its support. The following result shows that GAPM (nontrivially) partially identifies the expected log production function.

Proposition 2. *Suppose Assumptions 1-3 hold, then GAPM is refutable and the identified set may be nontrivial.*

Proof. GAPM implies that for all $s, t \in \mathcal{T}$, the following inequality holds

$$F_{js} - F_{jt} \leq \frac{1}{P_{jt} e^{\epsilon_{jt}}} \left[w_{jt}^1 (h_{js}^1 - h_{jt}^1) + w_{jt}^2 (h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) \right].$$

Dividing by F_{jt} on both sides, I obtain

$$\frac{F_{js}}{F_{jt}} \leq 1 + \frac{1}{P_{jt} W_{jt}} \left[w_{jt}^1 (h_{js}^1 - h_{jt}^1) + w_{jt}^2 (h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) \right],$$

where I used the equality $W_{jt} = F_{jt} e^{\epsilon_{jt}}$. Using Lemma 1, I can rewrite the inequality as

$$\frac{F_{js}}{F_{jt}} \leq 1 + \frac{RTS_j}{E_{jt}} \left[w_{jt}^1 (h_{js}^1 - h_{jt}^1) + w_{jt}^2 (h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) \right].$$

Observe that the left-hand side is always strictly positive. Further, note that wages and inputs can be such that $\frac{RTS_j}{E_{jt}} w_{jt}^1 (h_{js}^1 - h_{jt}^1) + w_{jt}^2 (h_{js}^1 - h_{jt}^2) + (c_{js} - c_{jt}) < 0$. For example, $w_{jt}^1 = h_{jt}^1 = c_{jt} = 10$, $w_{jt}^2 = h_{jt}^2 = 5$ and $w_{js}^1 = h_{js}^1 = c_{js} = 5$,

$w_{js}^2 = h_{js}^2 = 10$ makes it negative for both $s, t \in \mathcal{T}$. Hence, for any $RTS_j \in (0, 1]$ the following inequalities must hold simultaneously

$$\frac{F_{js}}{F_{jt}} < 1; \quad \frac{F_{jt}}{F_{js}} < 1.$$

Note that this implies $F_{js} < F_{jt}$ and $F_{jt} < F_{js}$, a contradiction. Next, consider wages and inputs $w_{jt}^1 = 2$, $w_{jt}^2 = 4$, $h_{jt}^1 = h_{jt}^2 = 15$, $c_{jt} = 10$ and $w_{js}^1 = w_{js}^2 = 2$, $h_{js}^1 = 5$, $h_{js}^2 = 10$, $c_{js} = 20$. Plugging those numbers in GAPM yields

$$\frac{F_{js}}{F_{jt}} \leq 1 - 0.3RTS_j$$
$$\frac{F_{jt}}{F_{js}} \leq 1 + 0.4RTS_j.$$

It is easy to see that these inequalities are only satisfied when $RTS_j \leq \frac{5}{6}$. In other words, GAPM may give upper bounds on returns to scale. Since this is true for every household, the support of the expected returns to scale can also be restricted. By Proposition 1, it follows that the identified set is nontrivial. □

The careful reader will have noticed that the proof of Proposition 2 only shows that GAPM can provide upper bounds on returns to scale. Thus, one may wonder whether GARP can provide an additional source of identification. The following result gives a negative answer.

Proposition 3. *Suppose Assumptions 1-3 hold, then GARP is not refutable and does not provide restrictions on the expected log production function.*

Proof. I want to show that GARP imposes no restrictions on returns to scale. In other words, I wish to show that there exist personalized prices for public expenditure $\mathcal{P}_{jt}^i$, personalized prices for children's welfare P_{jt}^i, and children's welfare W_{jt} such that any data set is consistent with every decreasing returns to scale. Recall that $a_{jst}^i = w_{jt}^i(l_{js}^i - l_{jt}^i) + (q_{js}^i - q_{jt}^i) + \mathcal{P}_{jt}^i(Q_{js} - Q_{jt}) + P_{jt}^i(W_{js} - W_{jt})$ and let $X_{jst}^i(\mathcal{P}_{jt}^i) := w_{jt}^i(l_{js}^i - l_{jt}^i) + (q_{js}^i - q_{jt}^i) + \mathcal{P}_{jt}^i(Q_{js} - Q_{jt})$. Lemma 1 implies $P_{jt}^i(W_{js} - W_{jt}) = RTS_j^{-1}(E_{js}\frac{P_{jt}^i}{P_{js}} - E_{jt}\frac{P_{jt}^i}{P_{jt}})$, where $E_{jt} := w_{jt}^1 h_{jt}^1 + w_{jt}^2 h_{jt}^2 + c_{jt}$. Fix $RTS_j \in (0, 1]$, $\mathcal{P}_{jt}^i$ to any number that satisfies $\mathcal{P}_{jt}^1 + \mathcal{P}_{jt}^2 = 1$, and $P_{jt}^1 = P_{jt}^2 = 0.5P_{jt}$

for all $t \in \mathcal{T}$. Next, pick $P_{jT} = 1$ and successively choose

$$0 < P_{jt} < \max_{i \in \{1,2\}, t' > t} \left\{ \left| -\frac{E_{jt} P_{jt'}}{2RTS_j X^i_{jtt'} - E_{jt'}} \right| \right\} \qquad t = T-1, T-2, \ldots, 1.$$

Observe that $a^i_{jtt'} > 0$ for every $t < t'$ such that a bundle from an earlier period is never revealed preferred to a bundle from a latter period. Hence, $a^i_{jtt'} < 0$ may only arise when $t > t'$, but then the construction ensures there cannot be a violation of GARP for either household member. Since the choice of RTS_j was arbitrary, GARP imposes no additional restrictions on returns to scale and, as such, no restrictions on the expected log production function. □

Proposition 3 implies that the model does not give lower bounds on returns to scale. From Proposition 1, it follows that I cannot obtain lower bounds on the slope of the expected log production function.[14] This implies, for example, that I cannot reject that the expected log production function is a constant function. The proof of Proposition 3 reveals that the problem is the unboundedness of relative personalized prices for children's welfare.[15] The next result shows that GARP can bound returns to scale if the support of personalized prices for children's welfare is compact.

Proposition 4. *Suppose Assumptions 1-3 hold and the support of personalized prices for children's welfare is compact, then GARP may restrict the support of the expected returns to scale such that the identified set is nontrivial.*

Proof. If the support of personalized prices for children's welfare is compact, then the ratio of prices across time periods is bounded. That is, $\frac{P_{jt}}{P_{js}} \in [\underline{P}, \overline{P}]$ for some $0 < \underline{P} \leq \overline{P}$. I need to show that GARP can provide lower and upper bounds on returns to scale under such support constraint. For the sake of concreteness, I

[14]It is possible to show that the result also applies with a parametric production function since productivity shocks are unrestricted.

[15]Proposition 3 is closely related to the classic non-refutability results of Varian (1988), who showed that when either a price or a quantity is unobserved, any finite dataset can be rationalized by appropriate choices of the unobserved variable; see also the correction by van Bruggen (2016). The mechanism here is slightly different, as both children's welfare and its price are unobservable, yet their product is restricted via the homogeneity of the production function.

assume that the support constraint is given by $\frac{P_{jt}}{P_{js}} \in [\frac{9}{10}, 1]$. The argument I lay out would apply for a larger support constraint given appropriate scaling of the data. It is worth noting, however, that a larger support constraint reduces the empirical content of GARP.

(Upper bound) Let $w^i_{jt} = 10$, $w^i_{js} = 5$, $l^i_{jt} = 20$, $l^i_{js} = 10$, $h^i_{jt} = 2$, $h^i_{js} = 4$, $q_{jt} = 40$, $q_{js} = 100$, $c_{jt} = 10$, $c_{js} = 20$ and, for simplicity, $Q_{js} = Q_{jt}$. The latter is not crucial for my argument since $\mathcal{P}^i_{jt} \in (0, 1)$. Note that $X^i_{jst} = -40$, $X^i_{jts} = -10$, $E_{jt} = 60$, and $E_{js} = 50$. Further note that $a^1_{jst} < 0$ if and only if $a^2_{jst} < 0$ such that if either holds then $a_{jst} := a^1_{jst} + a^2_{jst} < 0$. Hence, if GARP holds for each household member GARP must also hold for the aggregate revealed preference relation a_{jst}. Note that $a_{jst} = -80 + P_{jt}(W_{js} - W_{jt})$ and $a_{jts} = -20 + P_{js}(W_{jt} - W_{js})$. By Lemma 1, I can express personalized prices for children's welfare as

$$W_{jt} = \frac{E_{jt}}{RTS_j P_{jt}}.$$

Hence, I can write the aggregate revealed preference relation as

$$\begin{aligned} a_{jst} &= -80 + RTS_j^{-1}\left(E_{js}\frac{P_{jt}}{P_{js}} - E_{jt}\right) \\ a_{jts} &= -20 + RTS_j^{-1}\left(E_{jt}\frac{P_{js}}{P_{jt}} - E_{js}\right). \end{aligned}$$

Observe that $a_{jst} < 0$ for all $RTS_j \in (0, 1]$. Thus, GARP may only be satisfied if $a_{jts} > 0$, which happens when $RTS_j < 5/6$.

(Lower bound) Let $w^i_{jt} = 8$, $w^i_{js} = 5$, $l^i_{jt} = 20$, $l^i_{js} = 10$, $h^i_{jt} = 3$, $h^i_{js} = 2$, $q_{jt} = 40$, $q_{js} = 150$, $c_{jt} = 40$, $c_{js} = 10$ and, for simplicity, $Q_{js} = Q_{jt}$. Note that $X^i_{jst} = 30$, $X^i_{jts} = -60$, $E_{jt} = 88$, and $E_{js} = 30$. Similar to before, I can write the aggregate revealed preference relation as

$$\begin{aligned} a_{jst} &= 60 + RTS_j^{-1}\left(30\frac{P_{jt}}{P_{js}} - 88\right) \\ a_{jts} &= -120 + RTS_j^{-1}\left(88\frac{P_{js}}{P_{jt}} - 30\right). \end{aligned}$$

Observe that $a_{jts} < 0$ for all $RTS_j \in (0, 1]$. Thus, GARP may only be satisfied if $a_{jst} > 0$, which happens when $RTS > \frac{41}{45}$.

For the sake of simplicity, suppose every household has the same data set. Then, the support of $\mathbb{E}[RTS]$ is a strict subset of $(0, 1]$ such that the identified set is nontrivial. $\square$

To gain some intuition on the mechanism by which household members' choices provide information on RTS, consider the case where $X^i_{t_j,t_k} > 0$ and $X^i_{t_k,t_j} > 0$. Without loss of generality, suppose $P^i_{t_k}(W_{t_j} - W_{t_k}) \geq 0$ such that $P^i_{t_j}(W_{t_k} - W_{t_j}) \leq 0$. Further suppose $a^i_{t_k,t_j} \leq 0$ such that an informative upper bound is obtained from the data. Since $a^i_{t_k,t_j} \leq 0$, household members prefer the allocation $(l^i_{t_j}, q^i_{t_j}, Q_{t_j}, W_{t_j})$ over $(l^i_{t_k}, q^i_{t_k}, Q_{t_k}, W_{t_k})$ in period t_j. Since $a^i_{t_k,t_j} \leq 0$ despite $X^i_{t_k,t_j} > 0$, it must be that $(l^i_{t_j}, q^i_{t_j}, Q_{t_j}, W_{t_j})$ is preferred to $(l^i_{t_k}, q^i_{t_k}, Q_{t_k}, W_{t_k})$ because children's welfare is sufficiently more enticing. Children's welfare is more enticing if it gives a higher marginal utility or when $P^i_{t_j}$ is large, but this exactly occurs when RTS is not too large.

Remark. In practice, it may be necessary to bound the support of the latent variables such as in the application of this paper. In such case, GARP naturally provides meaningful though possibly mild restrictions on the production technology. As Proposition 4 makes clear, the analyst can obtain stronger restrictions on the production technology if prior knowledge about the support of the latent variables is available.

4 Empirical Specification

The previous section showed that the model nonparameterically partially identifies the production function. To improve the interpretability of the empirical results, I now specialize the production function to a Cobb-Douglas technology.[16]

[16] The empirical human capital literature often finds that the production function is well-approximated by a Cobb-Douglas specification (see, e.g., Attanasio, Meghir and Nix, 2020; Attanasio et al., 2020a,b). Using the same data as in my application, Cherchye, De Rock and Vermeulen (2012) estimate an elasticity of substitution close to one. In line with their findings, I cannot reject the Douglas-Douglas specification in my empirical analysis.

Assumption 4. *The production function is Cobb-Douglas such that*

$$W_{jt} = (h_{jt}^1)^{\alpha_{j1}} (h_{jt}^2)^{\alpha_{j2}} (c_{jt})^{\alpha_{j3}} e^{\epsilon_{jt}}.$$

The Cobb-Douglas technology is a natural choice as it is homogeneous of degree $RTS = \alpha_{j1} + \alpha_{j2} + \alpha_{j3}$. Furthermore, it is easy to see that the output elasticities are given by

$$\begin{aligned}
\alpha_{j1} &= \frac{RTS_j w_{jt}^1 h_{jt}^1}{w_{jt} h_{jt}^1 + w_{jt} h_{jt}^2 + c_{jt}} \\
\alpha_{j2} &= \frac{RTS_j w_{jt}^2 h_{jt}^2}{w_{jt} h_{jt}^1 + w_{jt} h_{jt}^2 + c_{jt}} \\
\alpha_{j3} &= \frac{RTS_j c_{jt}}{w_{jt} h_{jt}^1 + w_{jt} h_{jt}^2 + c_{jt}}.
\end{aligned}$$

In words, the model implies that each output elasticity equates a fraction RTS of its share of total investment on children. These shares are constant in time, regardless of changes in the shadow price of children's welfare, P_{jt}. The next result warns against ignoring productivity shocks in the model.

Claim 1. *Suppose Assumptions 1-4 hold. If productivity shocks are ignored, then the data may erroneously reject the model at the true returns to scale.*

Proof. In what follows, I remove the j subscript from the variables. Suppose the data are rationalized by the model at the true returns to scale $RTS^0 \in (0, 1]$ and the true children's welfare $W_t = (h_t^1)^{\alpha_1} (h_t^2)^{\alpha_2} (c_t)^{\alpha_3} e^{\epsilon_t}$. Suppose now the econometrician ignores productivity shocks and assumes

$$\widetilde{W}_t = (h_t^1)^{\alpha_1} (h_t^2)^{\alpha_2} (c_t)^{\alpha_3}.$$

Conditional on RTS^0, the output elasticities are identified. Therefore, children's welfare is also identified. From the first-order conditions of the model and Lemma 1, I have

$$\widetilde{P}_t = \frac{E_t}{RTS^0 \widetilde{W}_t}.$$

Since $\widetilde{W}_t$ is identified, it follows that $\widetilde{P}_t$ is also identified. It is then obvious that the Afriat inequalities

$$U_s^i - U_t^i \leq \lambda_t^i \left[w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + \widetilde{P}_t^i(\widetilde{W}_s - \widetilde{W}_t) \right]$$

can be rejected by the data even though the data would be consistent with the Afriat inequalities under the correct specification of the production function. □

Since output elasticities are a function of returns to scale, Claim 1 implies that ignoring productivity shocks may lead to inconsistent output elasticities.[17] The next result shows that the first-order conditions of the model have empirical bite under the Cobb-Douglas specification.

Claim 2. *Suppose Assumptions 1-4 hold. The first-order conditions of the model are refutable independently of returns to scale.*

Proof. By Lemma 1, I have

$$P_{jt}W_{jt} = RTS_j^{-1}(w_{jt}^1 h_{jt}^1 + w_{jt}^2 h_{jt}^2 + c_{jt}) \quad \forall t \in \mathcal{T},$$

where $RTS \in (0, 1]$. For the sake of simplicity, suppose there are only two time periods. As such, I have

$$\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} = \frac{(w_{j2}^1 h_{j2}^1 + w_{j2}^2 h_{j2}^2 + c_{j2})}{(w_{j1}^1 h_{j1}^1 + w_{j1}^2 h_{j1}^2 + c_{j1})}.$$

Since output elasticities are time invariant, it must be that the following set of equations holds

$$\begin{aligned}
\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} &= \frac{w_{j2}^1 h_{j2}^1}{w_{j1}^1 h_{j1}^1} \\
\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} &= \frac{w_{j2}^2 h_{j2}^2}{w_{j1}^2 h_{j1}^2} \\
\frac{P_{j2}W_{j2}}{P_{j1}W_{j1}} &= \frac{c_{j2}}{c_{j1}}.
\end{aligned}$$

[17] More generally, productivity shocks are useful as they may absorb omitted variables that could otherwise bias the output elasticities.

Note that these equations do not depend on returns to scale. Furthermore, they can easily be violated such as with $w^1_{j2}h^1_{t2} = 1/2$ and $w^2_{j2}h^2_{t2} = c_{j2} = 1/4$. $\square$

Claim 2 shows that the first-order conditions have meaningful implications that can be tested in the data given a Cobb-Douglas specification. The nonparametric results from the previous section further guarantee that returns to scale and thus the output elasticities are restricted.

4.1 Measurement Error

Claim 2 shows that the model implies a set of overidentifying restrictions on output elasticities. Consequently, any measurement error in the inputs, however small, would lead to the erroneous rejection of the model. It follows that any test of the model that does not address this issue would be dubious in this framework. For this reason, I allow for measurement error in inputs and impose mild centering conditions. Let $m^x_t := x_t - x^\star_t$ denote the difference between the observed and true value of a variable x_t in period t.

Assumption 5. $\mathbb{E}[m^x_t] = 0$, *where* $x \in \{h^1, h^2, c\}$, $t = 1, 2, \ldots, T$.

Assumption 5 requires that observed inputs be consistent with the true inputs on average in the cross-section, reflecting empirical evidence that mean expenditures are accurately reported in surveys (Kolsrud, Landais and Spinnewijn, 2017; Abildgren et al., 2018). These moment conditions do not impose any parametric form on the distribution of measurement error, nor do they require it to be identical over time. While the focus on inputs is motivated by the overidentifying restrictions, the framework is flexible enough to accommodate measurement error in other variables if needed.

5 Testing and Estimation

This section presents the statistical framework used for testing the model and making inference on the production function.

5.1 Testing

Let $\zeta_j := \{(U^i_{jt}, \lambda^i_{jt}, \mathcal{P}^i_{jt}, P^i_{jt}, W_{jt}, \boldsymbol{\alpha}_{jt}, \omega_{jt}, m^x_{jt})_{i\in\{1,2\}, x\in\{h^1,h^2,c\}}\}_{t\in\mathcal{T}} \in Z|\mathcal{D}$ denote the set of household-specific latent variables in the model, where Z denote the support of the latent variables and $\mathcal{D}$ denote the support of the data. The revealed preference characterization along with the moment conditions can be used to define the statistical rationalizability of a panel data set $D := \{D_j\}_{j\in\mathcal{J}}$. To this end, write the constraints of the model in the form of moment functions:

$$
\begin{aligned}
g^U_{ist}(D_j,\zeta_j) &:= \mathbb{1}\Bigg(U^i_{js} - U^i_{jt} \leq \lambda^i_{jt}\Big[w^i_{jt}(l^i_{js} - l^i_{jt}) + (q^i_{js} - q^i_{jt}) + \\
&\qquad + \mathcal{P}^i_{jt}(Q_{js} - Q_{jt}) + P^i_{jt}(W_{js} - W_{jt})\Big]\Bigg) - 1 \\
g^{\alpha_1}_t(D_j,\zeta_j) &:= \mathbb{1}\left(\alpha_{jt1} = \frac{w^1_{jt}h^1_{jt}}{P_{jt}W_{jt}}\right) - 1 \\
g^{\alpha_2}_t(D_j,\zeta_j) &:= \mathbb{1}\left(\alpha_{jt2} = \frac{w^2_{jt}h^2_{jt}}{P_{jt}W_{jt}}\right) - 1 \\
g^{\alpha_3}_t(D_j,\zeta_j) &:= \mathbb{1}\left(\alpha_{jt3} = \frac{c_{jt}}{P_{jt}W_{jt}}\right) - 1 \\
g^W_t(D_j,\zeta_j) &:= \mathbb{1}\left(W_{jt} = (h^1_{jt})^{\alpha_{j1}}(h^2_{jt})^{\alpha_{j2}}(c_{jt})^{\alpha_{j3}}e^{\epsilon_{jt}}\right) - 1 \\
g^m_{xt}(D_j,\zeta_j) &:= m^x_{jt},
\end{aligned}
$$

where $\mathbb{1}(\cdot)$ denote the indicator function and equates 1 if the expression inside the parenthesis is satisfied and 0 otherwise. The latent variables further need to satisfy their support constraints such that $\mathcal{P}^1_{jt} + \mathcal{P}^2_{jt} = 1$.

Note that I let the output elasticities vary over time in the moment functions $g^{\alpha_k}_t(D_j,\zeta_j)$, which allows the corresponding equations to be satisfied in every period. To impose time invariant output elasticities, I require their expected variance to be zero:

$$\mathbb{E}[\boldsymbol{g}^v(D_j,\zeta_j)] = 0,$$

where $\boldsymbol{g}^v(D_j,\zeta_j) := var(\boldsymbol{\alpha})$. Since the variance is always non-negative, those moment conditions are satisfied if and only if the variance is zero for all households. As such, this formulation is equivalent to directly imposing that production parameters are time invariant.

In what follows, let $\boldsymbol{g}(D_j, \zeta_j)$ denote the vector of all moment functions. The subset corresponding to measurement error and the variance of output elasticities is denoted by $\boldsymbol{g}^{(m,v)}(D_j, \zeta_j) := (\boldsymbol{g}^m(D_j, \zeta_j)', \boldsymbol{g}^v(D_j, \zeta_j)')'$, while its complement is denoted by $\boldsymbol{g}^{-(m,v)}(D_j, \zeta_j) := (\boldsymbol{g}^U(D_j, \zeta_j)', \boldsymbol{g}^\alpha(D_j, \zeta_j)', \boldsymbol{g}^W(D_j, \zeta_j)')'$.

Definition 8. Under Assumptions 1-5, a data set D is statistically rationalizable if

$$\inf_{\mu \in \mathcal{M}_{\mathcal{Z}|\mathcal{D}}} \|\mathbb{E}_{\mu \times \pi_0}[\boldsymbol{g}(D, \zeta)]\| = 0,$$

where $\mathcal{M}_{Z|\mathcal{D}}$ is the set of all conditional probability distributions on $Z|\mathcal{D}$ and $\pi_0 \in \mathcal{M}_{\mathcal{X}}$ is the observed distribution of D.

In its current form, the notion of statistical rationalizability has $2T^2 + T + T + T + T + 3T + T$ moment conditions, including some that are discontinuous. Let d_m denote the number of moment conditions on measurement error and d_v denote the number of moment conditions on the variance of the output elasticities. The following result due to Schennach (2014) and Aguiar and Kashaev (2021) allows us to considerably reduce the complexity of the problem.

Proposition 5. *Under Assumptions 1-5, a data set D is statistically rationalizable if and only if*

$$\min_{\boldsymbol{\gamma} \in \mathbb{R}^{d_m + d_v}} \|\mathbb{E}_{\pi_0}[\bar{\boldsymbol{g}}(D; \boldsymbol{\gamma})]\| = 0,$$

where

$$\bar{\boldsymbol{g}}_j(D_j; \boldsymbol{\gamma}) := \frac{\int_{\zeta_j \in Z|\mathcal{D}} \boldsymbol{g}_j^{(m,v)}(D_j, \zeta_j) \exp\left(\boldsymbol{\gamma}' \boldsymbol{g}_j^{(m,v)}(D_j, \zeta_j)\right) \mathbb{1}(\boldsymbol{g}_j^{-(m,v)}(D_j, \zeta_j) = 0)\, d\eta(\zeta_j|D_j)}{\int_{\zeta_j \in Z|\mathcal{D}} \exp\left(\boldsymbol{\gamma}' \boldsymbol{g}_j^{(m,v)}(D_j, \zeta_j)\right) \mathbb{1}(\boldsymbol{g}_j^{-(m,v)}(D_j, \zeta_j) = 0)\, d\eta(\zeta_j|D_j)},$$

and where $\eta(\cdot|D_j)$ is an arbitrary user-specified distribution supported on $Z|\mathcal{D}$ such that $\mathbb{E}_{\pi_0}[\log(\mathbb{E}_\eta[\exp(\boldsymbol{\gamma}' \boldsymbol{g}^{(m,v)}(D, \zeta))|D])]$ exists and is twice continuously differentiable in $\boldsymbol{\gamma}$ for all $\boldsymbol{\gamma} \in \mathbb{R}^{d_m + d_v}$.

The previous result calls for some comments. First, the dimensionality of the problem is greatly reduced as it only requires finding a finite dimensional parameter $\boldsymbol{\gamma}$ rather than a distribution μ. Second, the moment conditions associated with the concavity of the utility functions, first-order conditions, and production function equations are directly imposed on each household data set such as to restrict the support of the unobservables. In particular, observe that the optimization problem no longer includes any discontinuous moment function. Finally, it is worth noting that the result states that there is no loss in generality in averaging out the unobservables in the moment functions provided the distribution is from the exponential family.

The simplification allowed by Proposition 5 requires finding unobservables ζ_j that exactly satisfy the concavity of the utility functions, first-order conditions, and production function equations. If the constraints were linear in the unobservables, it would be possible to use a standard Hit-and-Run algorithm to directly sample them from the feasible space defined by the intersection of the inequalities and the system of equations. Unfortunately, the inequalities for the concavity of the utility functions are highly nonlinear because they involve multiplicative interactions between the unobservables. As a result, the feasible space is curved rather than polyhedral, making standard geometric sampling methods infeasible.[18]

I resolve this pervasive issue by proposing a blocked Gibbs sampler. The idea is to break down the sampling procedure into multiple blocks, where each block takes a subset of all unobservables as given. The key is to create those blocks in such a way that the inequalities are linear in the unobservables conditional on a certain subset of all unobservables. Thus, the inequalities effectively define a conditional convex polytope in each block. This allows for a straightforward sampling procedure that guarantees the unobservables to exactly satisfy the inequalities, first-order conditions, and production function equations. The details of the algorithm are provided in the Appendix.

[18] In principle, it would be possible to use rejection sampling along with a mixed-integer programming (MIP) problem to draw from the feasible space. However, these types of MIP for collective models are NP-complete (Nobibon et al., 2016) so they do not scale well. Also, rejection sampling is generally inefficient.

5.2 Inference

One of the advantages of ELVIS is that testing and inference are quite straightforward even if the model is partially identified. Indeed, testing the model can be done by constructing the sample analogues of the averaged moments along with their covariance, defined as

$$\hat{\bar{\boldsymbol{g}}}(\boldsymbol{\gamma}) := \frac{1}{J}\sum_{j=1}^{J}\bar{\boldsymbol{g}}_j(D_j,\boldsymbol{\gamma}), \quad \hat{\bar{\boldsymbol{\Omega}}}(\boldsymbol{\gamma}) := \frac{1}{J}\sum_{j=1}^{J}\bar{\boldsymbol{g}}_j(D_j,\boldsymbol{\gamma})\bar{\boldsymbol{g}}_j(D_j,\boldsymbol{\gamma})' - \hat{\bar{\boldsymbol{g}}}_j(\boldsymbol{\gamma})\hat{\bar{\boldsymbol{g}}}_j(\boldsymbol{\gamma})'.$$

The test statistic is then defined as

$$\mathrm{TS}_J := J \inf_{\boldsymbol{\gamma}\in\mathbb{R}^{d_m+d_v}} \hat{\bar{\boldsymbol{g}}}(\boldsymbol{\gamma})'\hat{\bar{\boldsymbol{\Omega}}}^{-}(\boldsymbol{\gamma})\hat{\bar{\boldsymbol{g}}}(\boldsymbol{\gamma}),$$

where $\hat{\bar{\boldsymbol{\Omega}}}^{-}$ denote the generalized inverse of the matrix $\hat{\bar{\boldsymbol{\Omega}}}$. Since the test statistic is stochastically bounded by the chi-square distribution, statistical rationalizability of the model amounts to comparing the value of the test statistic against the chi-square critical value with $d_m + d_v$ degrees of freedom at significance level α. Inference on parameters of interest $\boldsymbol{\theta}$ is conducted by adding d_θ moments and inverting the test statistic using the chi-square critical value with $d_m + d_v + d_\theta$ degrees of freedom at significance level α. Importantly, the identified set is convex under mild conditions, ensuring that it can be recovered efficiently.[19]

6 Data

I conduct my empirical analysis using the Longitudinal Internet Studies for the Social Sciences (LISS) panel, which follows approximately 5,000 Dutch households annually and is representative of the Dutch population. The survey provides detailed information on household expenditures and time use across several categories, allowing me to directly observe the private expenditures q_t^i of each adult household member. Time use is reported for the past seven days, and monthly ex-

[19] I refer the reader to Aguiar and Kashaev (2021) for additional details about the statistical procedure.

penditures are self-reported, which may introduce some measurement error. This is addressed explicitly in Section 4.

The analysis is restricted to couples with children, reducing the sample to roughly 1,000 households. I further restrict the sample to households with non-missing and nonzero wages, private expenditures, and public expenditures. Missing and zero values in the inputs of the production function are imputed using the sample mean by year, where the treatment of measurement error explicitly handles the imperfection of the imputation. Finally, I only retain households appearing in the panel for three periods to maintain a balanced panel while exploiting panel variation.

The final sample consists of 132 couples with children, observed over three periods pooled from 2008 to 2017.[20] While relatively small, this sample size is comparable to Cherchye, De Rock and Vermeulen (2012) despite my restriction to panel data. Summary statistics for the sample are presented in Table 1, and further details on the sample construction are provided in the Appendix.

Table 1: Sample Summary Statistics

	Husband		Wife		Household	
	Mean	Std dev.	Mean	Std dev.	Mean	Std dev.
Age	46.25	7.99	44.08	7.28		
Wage (EUR/hour)	13.68	6.65	12.95	12.00		
Number of children					2.00	0.83
Mean age of children					13.15	6.36
Childcare (hours/week)	10.41	7.89	16.92	13.14		
Work (hours/week)	37.50	6.03	23.30	8.27		
Private expenditure (EUR/month)	362.00	229.23	395.81	323.30		
Public expenditure (EUR/month)					2164.85	836.12
Expenditure on children (EUR/month)					512.52	512.90
Total households						132

Table 1 illustrate the gendered division of labor and provide context for household resource allocation in the sample. On average, husbands are slightly older than wives (46.3 vs 44.1 years) and earn slightly higher hourly wages (13.68 vs

[20]Ideally, a larger sample would be preferable for the asymptotic theory. Future research could expand the sample by including households with missing wage data.

12.95 euros). Households have, on average, two children with a mean age of 13 years. Regarding time use, wives spend more hours on childcare than husbands (16.9 vs 10.4 hours per week), while husbands spend more time on paid work (37.5 vs 23.3 hours per week). Finally, average monthly private expenditures are broadly similar between spouses.

7 Empirical Results

This section recovers confidence sets on expected returns to scale and expected output elasticities. Then, it investigate how the production function changes with demographics.

7.1 Expected Parameters

I begin the empirical analysis by recovering the 95% confidence set for the expected returns to scale. Because the confidence set is convex, it is sufficient to recover its lower and upper bounds. Doing so, I find that the 95% confidence set for the expected returns to scale is $[0.270, 0.405]$.[21] Hence, a 10% increase in all inputs would increase children's welfare by about 3 to 4 percent. Next, I proceed to recover 95% confidence sets for the expected output elasticities. These results are presented in Figure 1.

The figure show that, on average, time inputs by mothers increases children's welfare by more than time inputs by fathers or expenditure on children. More precisely, a 10% increase in childcare by the mother would increase children's welfare by about 1.4% while a 10% increase in childcare by the father would increase children's welfare by about 1%. The impact of expenditure on children appears lowest, where a 10% increase in expenditure on children would only increase children's welfare by about 0.9%.

[21]Since the confidence set is nonempty, it follows that the model is not rejected by the data. In particular, this implies that the Cobb-Douglas specification of the production function is not rejected by the data.

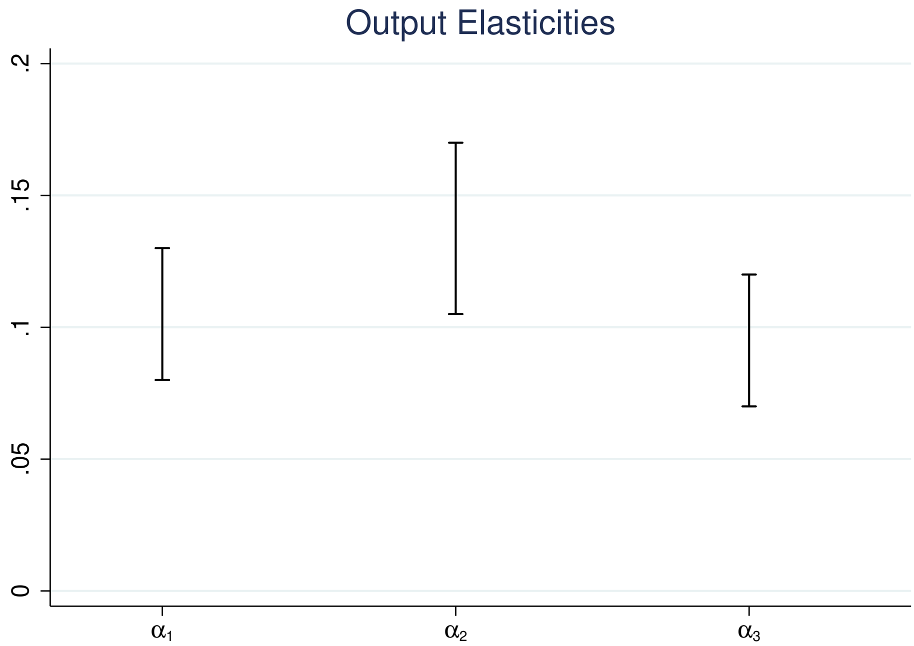

Figure 1: 95% Confidence Sets on Production Parameters

7.2 Heterogeneity

This section investigates how the expected production technology varies with household characteristics. Due to the small sample size, I choose to the analysis using linear regressions. First, I analyze heterogeneity in returns to scale through the regression

$$\boldsymbol{RTS} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\omega}, \tag{2}$$

where $\boldsymbol{RTS} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3$ represents returns to scale, $\boldsymbol{X}$ is a set of covariates, and $\boldsymbol{\omega}$ is a random error. Likewise, I analyze heterogeneity in output elasticities through the regressions

$$\boldsymbol{\alpha}_k = \boldsymbol{X}\widetilde{\boldsymbol{\beta}}_k + \widetilde{\boldsymbol{\omega}}_k, \tag{3}$$

where $\boldsymbol{\alpha}_k$ represents output elasticities with respect to input k, $\boldsymbol{X}$ is a set of covariates, and $\widetilde{\boldsymbol{\omega}}_k$ is a random error. In each specification, I assume that the error term has conditional mean zero.[22]

[22]In the results of Table 2, I impose and test $\mathbb{E}[X_k\omega] = 0$, where k denote the kth covariate

I wish to emphasize that these regression equations are not estimated separately from the rest of the model, but rather imposed as additional equation restrictions within the model. That is, I make inference on the expected parameters of a regression by adding a moment function such as

$$g^{\beta}(D_j, \zeta_j) := \mathbb{1}(RTS_j = \boldsymbol{X}_j \boldsymbol{\beta} + \omega_j).$$

Since the output elasticities are estimated jointly with their dependence on demographic characteristics, the residual in the regression captures household-specific differences in elasticities that are not explained by demographics, and inference properly accounts for the uncertainty in the estimated elasticities.

The 95% confidence sets on the expected coefficients of the regressions are reported in Table 2.

Table 2: 95% Confidence Sets on Regression Coefficients

	Dependent Variable			
Independent Variable[a]	$\alpha_1 + \alpha_2 + \alpha_3$	α_1	α_2	α_3
HAVO, VWO & MBO (Father)	$[-0.07, 0.05]$	$[0.02, 0.04]$	$[-0.03, 0.00]$	$[-0.08, 0.03]$
HBO & WO (Father)	$[-0.12, 0.02]$	$[0.00, 0.02]$	$[-0.04, -0.03]$	$[-0.08, 0.02]$
HAVO, VWO & MBO (Mother)	$[0.06, 0.20]$	$[0.02, 0.05]$	$[0.05, 0.08]$	$[-0.02, 0.10]$
HBO & WO (Mother)	$[0.02, 0.15]$	$[0.01, 0.03]$	$[0.04, 0.07]$	$[-0.05, 0.08]$
#Children	$[-0.06, -0.01]$	$[-0.03, -0.02]$	$[-0.03, -0.01]$	$[-0.01, 0.03]$
Age Children	$[0.00, 0.01]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.01]$
Dwelling	$[-0.33, -0.10]$	$[-0.08, -0.06]$	$[-0.12, -0.10]$	$[-0.14, 0.08]$

[a] The education category "HAVO, VWO, & MBO" represents general education that leads to higher education and vocational education that can lead to higher education. The education category "HBO & WO" represents higher education. #children is the number of children in the household. Age Children is the average age of children in the household. Dwelling is an indicator that takes value 0 if the household rents and 1 if it owns a house.

The first four rows of Table 2 report the impacts of education on the production technology. Education is treated as a categorical variable reflecting the highest level of education attained in the Netherlands. The reference category corresponds to primary school and pre-vocational secondary education (VMBO). The second category includes general education leading to higher education (HAVO and VWO) and vocational education that can lead to higher education (MBO). The third category represents higher education (HBO and WO). This classification

in the regression. The nonrejection of the restriction provides support for the conditional mean zero assumption.

reflects the typical educational pathways in the Dutch system.

Table 2 shows that higher education increases the expected output elasticity by roughly 0.02 for fathers and 0.06 for mothers. A positive spillover exists from mothers, with higher maternal education boosting fathers' expected output elasticity, whereas paternal education has no comparable effect on mothers. Paternal education does not appear to affect expected returns to scale, while maternal education increases it by about 0.10. Finally, the impacts of education on the expected output elasticity with respect to expenditure on children are inconclusive, as the confidence sets are wide and include zero.

Table 2 also shows that the number of children in the household has a small but negative effect on expected returns to scale and on expected output elasticities with respect to time inputs. More notably, household dwelling has a substantial negative impact on the production of children's welfare, relative to home ownership. For instance, renting households have expected output elasticities with respect to time inputs lower by about 0.07 for fathers and 0.11 for mothers. Furthermore, based on the 95% confidence set, expected returns to scale for households that do not own their home are estimated to decrease by 0.10 to 0.33.

8 Conclusion

This paper proposes a novel framework to evaluate the impacts of parental inputs on children's welfare within a collective model that allows for heterogeneity in preferences and production technology. I show that conditions for point identification in collective models with children, such as constant returns to scale, are rejected by the data. Instead, my findings are consistent with the empirical human capital literature, such as in Del Boca, Flinn and Wiswall (2014), where the impacts of parental inputs on child development exhibit decreasing returns to scale. Although my results provide support for the collective model approach to analyzing child development, they also warn against assuming constant returns to scale in the production technology. Indeed, child human capital development is now recognized as an important factor in assessing the costs and benefits of welfare reforms (Mullins, 2022), but such cost-benefit analysis crucially depends on the shape of the production technology. Furthermore, my empirical results show that the education level of the mother and the household environment play important roles in the development of children. Those findings provide support for policy interventions targeted at disadvantaged households such as to mitigate children's achievement gaps. In spite of the advances made in this paper, there is a need for future research along multiple directions. First, an analysis of child development that allows for potential complementarities in inputs within the collective model may reveal additional interesting patterns in the production of child human capital. Second, additional data on time use such as active and passive time spent with children may provide insights into the reasons for differences in parenting skills between household demographics. Third, an extension of the collective model with children to a dynamic setup would enable one to quantify how early investments in children impact returns to later investments.

Appendix

A Proofs

A.1 Proof of Theorem 1

$(i) \implies (ii)$

The household problem can be written as

$$\max_{(l^1,l^2,h^1,h^2,q^1,q^2,Q,c)\in\mathbb{R}^2_+\times\mathbb{R}^2_{++}\times\mathbb{R}^2_+\times\mathbb{R}_+\times\mathbb{R}_{++}} \mu_t^1 U^1(l^1,q^1,Q,W) + \mu_t^2 U^2(l^2,q^2,Q,W),$$

subject to satisfying the household constraints

$$\begin{aligned}(q^1+q^2)+Q+c &= w_t^1(\tau-l^1-h^1)+w_t^2(\tau-l^2-h^2)\\ W &= F(h^1,h^2,c)e^{\epsilon_t}.\end{aligned}$$

The first-order conditions are given by

$$\begin{aligned}\mu_t^i\frac{\partial U^i}{\partial l^i} &= \eta_t w_t^i\\ \mu_t^i\frac{\partial U^i}{\partial q^i} &= \eta_t.\\ \sum_i \mu_t^i\frac{\partial U^i}{\partial W_t}\cdot\frac{\partial W_t}{\partial h^1} &= \eta_t w_t^1\\ \sum_i \mu_t^i\frac{\partial U^i}{\partial W_t}\cdot\frac{\partial W_t}{\partial h^2} &= \eta_t w_t^2\\ \sum_i \mu_t^i\frac{\partial U^i}{\partial W_t}\cdot\frac{\partial W_t}{\partial c} &= \eta_t\\ \sum_i \mu_t^i\frac{\partial U^i}{\partial Q} &= \eta_t,\end{aligned}$$

where the equalities hold for some supergradient of the utility function.[23] Next,

[23]For corner solutions, the first-order conditions may only hold with inequality. The argument does not require any substantive change to accommodate this possibility.

define

$$\lambda_t^i = \frac{\eta_t}{\mu_t^i}$$
$$P_t^i = \frac{\mu_t^i}{\eta_t}\frac{\partial U^i}{\partial W_t}$$
$$\mathcal{P}_t^i = \frac{\mu_t^i}{\eta_t}\frac{\partial U^i}{\partial Q}.$$

The first-order conditions can be rewritten as

$$\frac{\partial U^i}{\partial l^i} = \lambda_t^i w_t^i \tag{4}$$
$$\frac{\partial U^i}{\partial q^i} = \lambda_t^i. \tag{5}$$
$$(P_t^1 + P_t^2)\frac{\partial W_t}{\partial h^1} = w_t^1 \tag{6}$$
$$(P_t^1 + P_t^2)\frac{\partial W_t}{\partial h^2} = w_t^2 \tag{7}$$
$$(P_t^1 + P_t^2)\frac{\partial W_t}{\partial c} = 1 \tag{8}$$
$$\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1. \tag{9}$$

Using the concavity of the utility functions I obtain

$$U_s^i - U_t^i \leq \left[\frac{\partial U^i}{\partial l_t^i}(l_s^i - l_t^i) + \frac{\partial U^i}{\partial q_t^i}(q_s^i - q_t^i) + \frac{\partial U^i}{\partial Q}(Q_s - Q_t) + \frac{\partial U^i}{\partial W_t}(W_s - W_t)\right],$$

where $U_t^i := U^i(l_t^i, q_t^i, Q_t, W_t)$ for all $t \in \mathcal{T}$. Substituting the derivatives of the utility function for their expressions yields

$$U_s^i - U_t^i \leq \lambda_t^i\Big[w_t^i(l_s^i - l_t^i) + (q_s^i - q_t^i) + \mathcal{P}_t^i(Q_s - Q_t) + P_t^i(W_s - W_t)\Big].$$

Next, using the concavity of the production function I obtain

$$F_s - F_t \leq \frac{\partial F}{\partial h_t^1}(h_s^1 - h_t^1) + \frac{\partial F}{\partial h_t^2}(h_s^2 - h_t^2) + \frac{\partial F}{\partial c_t}(c_s - c_t),$$

where $F_t := F(h_t^1, h_t^2, c_t)$ for all $t \in \mathcal{T}$. Substituting the derivatives of the production function from equations (6)-(8) yields

$$F_s - F_t \le \frac{w_t^1}{P_t e^{\epsilon_t}}(h_s^1 - h_t^1) + \frac{w_t^2}{P_t e^{\epsilon_t}}(h_s^2 - h_t^2) + \frac{1}{P_t e^{\epsilon_t}}(c_s - c_t).$$

Putting everything together, these inequalities should hold for some U_t^i, $\lambda_t^i > 0$, $\mathcal{P}_t^i > 0$ such that $\mathcal{P}_t^1 + \mathcal{P}_t^2 = 1$, $P_t^i > 0$ such that $P_t^1 + P_t^2 = P_t$, W_t, $F_t > 0$ and ϵ_t such that $W_t = F_t e^{\epsilon_t}$, $t = 1, \dots, T$.

$(ii) \implies (i)$

I have to show that, if Theorem 1 (ii) holds, then there exist concave utility functions and a concave production function that rationalize the data. Thus, let $\tau = \{t_j\}_{j=1}^m$, $m \ge 2$, $t_j \in \mathcal{T}$ denote a sequence of indices and $\mathcal{I}$ denote the set of all such indices. Define

$$\begin{aligned}
&U^i(l^i, q^i, Q, W) := \\
&\quad \min_{\tau \in \mathcal{I}} \Big\{ \lambda_{t_m}^i \Big[w_{t_m}^i (l^i - l_{t_m}^i) + (q^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i (Q - Q_{t_m}) + P_{t_m}^i (W - W_{t_m}) \Big] + \\
&\quad + \sum_{j=1}^{m-1} \lambda_{t_j}^i \Big[w_{t_j}^i (l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i (Q_{t_{j+1}} - Q_{t_j}) + P_{t_j}^i (W_{t_{j+1}} - W_{t_j}) \Big] \Big\}.
\end{aligned}$$

The function is the pointwise minimum of a collection of linear functions. Thus, it is continuous, increasing, and concave. By definition of U^i, there is some sequence of indices such that

$$\begin{aligned}
&U^i(l_t^i, q_t^i, Q_t, W_t) \ge \\
&\quad \lambda_{t_m}^i \Big[w_{t_m}^i (l_t^i - l_{t_m}^i) + (q_t^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i (Q_t - Q_{t_m}) + P_{t_m}^i (W_t - W_{t_m}) \Big] + \\
&\quad + \sum_{j=1}^{m-1} \lambda_{t_j}^i \Big[w_{t_j}^i (l_{t_{j+1}}^i - l_{t_j}^i) + (q_{t_{j+1}}^i - q_{t_j}^i) + \mathcal{P}_{t_j}^i (Q_{t_{j+1}} - Q_{t_j}) + P_{t_j}^i (W_{t_{j+1}} - W_{t_j}) \Big].
\end{aligned}$$

Add any allocation (l^i, q^i, Q, W) to the sequence and use the definition of U^i once again to obtain

$$\begin{aligned}
&\lambda_t^i \Big[w_t^i (l^i - l_t^i) + (q^i - q_t^i) + \mathcal{P}_t^i (Q - Q_t) + P_t^i (W - W_t) \Big] + \\
&+ \lambda_{t_m}^i \Big[w_{t_m}^i (l_t^i - l_{t_m}^i) + (q_t^i - q_{t_m}^i) + \mathcal{P}_{t_m}^i (Q_t - Q_{t_m}) + P_{t_m}^i (W_t - W_{t_m}) \Big] +
\end{aligned}$$

$$+\sum_{j=1}^{m-1}\lambda^i_{t_j}\Big[w^i_{t_j}(l^i_{t_{j+1}}-l^i_{t_j})+(q^i_{t_{j+1}}-q^i_{t_j})+\mathcal{P}^i_{t_j}(Q_{j+1}-Q_{t_j})+P^i_{t_j}(W_{t_{j+1}}-W_{t_j})\Big]$$
$$\geq U^i(l^i,q^i,W,Q).$$

Hence, rearranging the previous expression yields

$$U^i(l^i,q^i,Q,W)-U^i(l^i_t,q^i_t,Q_t,W_t)\leq\lambda^i_t\Big[w^i_t(l^i-l^i_t)+(q^i-q^i_t)+\mathcal{P}^i_t(Q-Q_t)+P^i_t(W-W_t)\Big].$$

Note that the two first supergradients of $U^i(l^i_t,q^i_t,Q_t,W_t)$ give the first-order conditions (4)-(5). Next, define

$$F(h^1,h^2,c):=\min_{\tau\in\mathcal{I}}\Big\{\frac{1}{P_te^{\epsilon_t}}\Big[w^1_t(h^1-h^1_t)+w^2_t(h^2-h^2_t)+(c-c_t)+$$
$$+\sum_{j=1}^{m-1}\Big[\frac{1}{P_{t_j}e^{\epsilon_{t_j}}}\Big[w^1_{t_j}(h^1_{t_{j+1}}-h^1_{t_j})+w^2_{t_j}(h^2_{j+1}-h^2_{t_j})+(c_{t_{j+1}}-c_{t_j})\Big]\Big]\Big\}.$$

This function is continuous, increasing, and concave in (h^1,h^2,c). By an identical argument as before, I obtain

$$F(h^1,h^2,c)-F(h^1_t,h^2_t,c_t)\leq\frac{1}{P_te^{\epsilon_t}}\Big[w^1_t(h^1-h^1_t)+w^2_t(h^2-h^2_t)+P^i_t(c-c_t)\Big].$$

Hence, the supergradients of $F(h^1_t,h^2_t,c_t)$ yield equations (6)-(8) and I conclude that Theorem 1 (*ii*) has the same implications as the household problem (1).

(*ii*) $\implies$ (*iii*)

Let us begin by noting that the Afriat inequalities can be combined such that for all $\{t_k\}_{k=1}^m\in\mathcal{I}$ and all $i\in\{1,2\}$

$$0\leq\sum_{k=1}^m\lambda^i_{t_{k+1}}a^i_{t_k,t_{k+1}}.$$

Observe that the set of all sequences $\mathcal{I}$ can be reduced to the set of all finite sequences as any sequence that satisfies this inequality is also satisfied without cycles. For the sake of a contradiction, suppose GARP is not satisfied for some household member. Then, there exists a cycle such that $a^i_{t_1,t_2}\leq 0$, $a^i_{t_2,t_3}\leq 0$, ...,

$a^i_{t_m,t_1} < 0$. Thus, it follows that

$$\lambda^i_{t_2} a^i_{t_1,t_2} + \lambda^i_{t_3} a^i_{t_2,t_3} + \cdots + \lambda^i_{t_1} a^i_{t_m,t_1} < 0,$$

a contradiction of cyclical monotonicity. Next, wish to show that GAPM holds. Observe that the inequalities for the production function can be rearranged as

$$P_t F_t e^{\epsilon_t} + w^1_t h^1_t + w^2_t h^2_t + c_t \leq P_t F_s e^{\epsilon_t} + w^1_t h^1_s + w^2_t h^2_s + c_s \quad \forall s, t \in \mathcal{T},$$

where I further have $W_t = F_t e^{\epsilon_t}$ by assumption.

$(iii) \implies (ii)$

Suppose that GARP holds for each household member. Then, an application of Fostel, Scarf and Todd (2004) shows the existence of the Afriat inequalities for each household member. Furthermore, rearranging the inequalities in GAPM yields the desired inequalities.

A.2 Proof of Lemma 1

Proof. From the first-order conditions of the model and the Hicks-neutrality of productivity shocks, I have

$$\begin{aligned}
\frac{\partial F(h^1_t, h^2_t, c_t)}{\partial h^1_t} e^{\epsilon_t} &= \frac{w^1_t}{P_t} \\
\frac{\partial F(h^1_t, h^2_t, c_t)}{\partial h^2_t} e^{\epsilon_t} &= \frac{w^2_t}{P_t} \\
\frac{\partial F(h^1_t, h^2_t, c_t)}{\partial c^2_t} e^{\epsilon_t} &= \frac{1}{P_t}.
\end{aligned}$$

I can multiply each marginal product by its own factor of production to get

$$\begin{aligned}
\frac{\partial F(h^1_t, h^2_t, c_t)}{\partial h^1_t} h^1_t e^{\epsilon_t} &= \frac{w^1_t h^1_t}{P_t} \\
\frac{\partial F(h^1_t, h^2_t, c_t)}{\partial h^2_t} h^2_t e^{\epsilon_t} &= \frac{w^2_t h^2_t}{P_t} \\
\frac{\partial F(h^1_t, h^2_t, c_t)}{\partial c^2_t} c_t e^{\epsilon_t} &= \frac{c_t}{P_t}.
\end{aligned}$$

Summing up these equations and multiplying by P_t, I obtain

$$P_t \left[\frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^1} h_t^1 + \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial h_t^2} h_t^2 + \frac{\partial F(h_t^1, h_t^2, c_t)}{\partial c_t^2} c_t \right] e^{\epsilon_t} = E_t,$$

where $E_t := w_t^1 h_t^1 + w_t^2 h_t^2 + c_t$. Since the production function is homogeneous of degree $RTS \in (0, 1]$, an application of Euler's theorem gives

$$RTS P_t W_t = E_t,$$

where I used the production function equation $W_t = F(h_t^1, h_t^2, c_t)e^{\epsilon_t}$. □

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